

Sine-square deformation and Möbius quantization of two-dimensional conformal field theory

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Abstract

Motivated by sine-square deformation (SSD) for quantum critical systems in 1+1-dimension, we discuss a Möbius quantization approach to the two-dimensional conformal field theory (CFT), which bridges the conventional radial quantization and the dipolar quantization recently proposed by Ishibashi and Tada. We then find that the continuous Virasoro algebra of the dipolar quantization can be interpreted as a continuum limit of the Virasoro algebra for scaled generators in the SSD limit of the Möbius quantization approach.

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I. INTRODUCTION

Recently, intriguing properties of sine-square deformation (SSD) for 1+1-dimensional quantum many-body systems have been revealed by a series of intensive theoretical researches.[1–13] The SSD was originally introduced as a spatial deformation of interaction couplings with the sine-square function for 1+1D quantum lattice models.[1–3] Several numerical studies clarified that critical ground states of SSD systems with the open-boundary are identical to those of the uniform systems with the periodic boundary within numerical accuracy. Later, this correspondence of the ground states between the uniform and SSD systems was proved for some exactly solved models which can be reduced to free fermionic models.[4, 5] In Ref.[6], an interesting connection of the SSD to supersymmetric quantum mechanics is pointed out for a nonrelativistic free fermion system in the continuous space. Moreover, the correspondence of the open and periodic boundary systems under the SSD has been successfully applied to numerical estimation of bulk quantities via finite-size systems[7–9]. These scalabilities of the SSD researches suggest that there exists rich physics behind the SSD.

The SSD of the two-dimensional conformal field theory (CFT)[14] was firstly investigated by Katsura, where the operator $L_0 - \frac{L_1+L_{-1}}{2} + \bar{L}_0 - \frac{\bar{L}_1+\bar{L}_{-1}}{2}$ is regarded as a Hamiltonian of the SSD system.[10] Here, L_n denotes the Virasoro generator of CFT. Then, an essential point is that the CFT vacuum is annihilated by the deformation terms $\frac{L_1+L_{-1}}{2}$ and $\frac{\bar{L}_1+\bar{L}_{-1}}{2}$, because of $SL(2,R)$ invariance, implying that the ground state of the SSD system is identical to that of the uniform system. Also, a relevance of the SSD to string theory was discussed on the basis of the free boson in Ref. [11]. Very recently, Ishibashi and Tada has proposed a more direct approach to the SSD of CFT, which is called “dipolar quantization”; this novel quantization scheme different from the conventional radial quantization provides a continuous Virasoro algebra for the SSD of CFT.[12, 13] However, the connection between the dipolar quantization and the radial quantization, or equivalently between the SSD system and the uniform system, has not been clear at the present stage of researches. How can we interpolate between the dipolar quantization for the SSD system and the usual CFT based on the radial quantization?

In order to address the above problem, we introduce the parametarized Hamiltonian that

bridges the uniform system and the SSD system[15],

$$\mathcal{H} = \mathcal{L}_0 + \bar{\mathcal{L}}_0 \quad (1)$$

with

$$\mathcal{L}_0 \equiv \cosh(2\theta)L_0 - \sinh(2\theta)\frac{L_1 + L_{-1}}{2}. \quad (2)$$

where we can assume $\theta \geq 0$ without loss of generality.[16] Note that $\theta = 0$ corresponds to the uniform system and $\theta \rightarrow \infty$ does to the SSD system except for the overall normalization. An essential observation of this Hamiltonian is that its ground state is always identical to that of the uniform system of $\theta = 0$, since Eq. (2) can be regarded as “Lorentz transformation” reflecting $SL(2, R)$ symmetry in CFT[11], and $\frac{L_1 + L_{-1}}{2}$ annihilates the CFT vacuum. In this paper, we derive the classical Virasoro (Witt) algebra based on the Hamiltonian (2), where the Möbius transformation coordinate plays a crucial role. We then discuss quantization of the Virasoro algebra for (1), which we call “Möbius quantization”, interpolating between the radial quantization and the dipolar quantization. In particular, we find that the continuous Virasoro algebra of the dipolar quantization can be interpreted as a continuum limit of the Virasoro algebra obtained in the SSD limit of the Möbius quantization. We also mention that the Möbius quantization approach gives a consistent result with the Möbius-type conformal mapping of CFT except at the SSD point.

This paper is organized as follows. In the next section, we briefly explain the classical Virasoro algebra and the role of the Möbius coordinate. In Sec. 3, we explain the Möbius quantization approach to the central extension of the classical Virasoro algebra. In particular, we derive the Virasoro algebra for Eq. (1) and discuss its continuum limit corresponding to the SSD point. We also describe the vacuum state and the primary field in the Möbius quantization. In Sec. 4, we comment on the conformal mapping associated with the Möbius quantization. In Sec. 5, we discuss the dipolar/SSD limit of the Möbius quantization in details. In Appendix A, the relation of the Virasoro generators between the Möbius quantization and the radial quantization is provided as a series expansion form, where expansion coefficients are exactly represented by Gauss’s hypergeometric function.

II. CLASSICAL VIRASORO ALGEBRA

In this section, let us discuss the classical Virasoro (Witt) algebra for Eq. (2). For the usual classical Virasoro algebra, the generator is defined as

$$l_n = -z^{-n+1} \frac{\partial}{\partial z}, \quad (3)$$

which satisfies the commutation relation

$$[l_n, l_{n'}] = (n - n') l_{n+n'}. \quad (4)$$

In analogy with Refs.[12, 13], we define a new differential operator for Eq. (2) as

$$\mathfrak{l}_0 \equiv \cosh(2\theta) l_0 - \sinh(2\theta) \frac{l_{-1} + l_1}{2} = -g(z) \frac{\partial}{\partial z}, \quad (5)$$

where

$$g(z) \equiv -\frac{1}{2} [\sinh(2\theta)(z^2 + 1) - 2 \cosh(2\theta)z]. \quad (6)$$

Note that $\theta = 0$ describes the uniform system and $\theta \rightarrow \infty$ corresponds to the SSD point. This definition of \mathfrak{l}_0 contains a divergent factor if $\theta \rightarrow \infty$. Thus, we also define a scaled generator

$$\tilde{\mathfrak{l}}_0 \equiv \frac{\mathfrak{l}_0}{N_\theta}, \quad (7)$$

where N_θ is the normalization factor,

$$N_\theta \equiv \cosh(2\theta). \quad (8)$$

It should be noted that the scaled generator $\tilde{\mathfrak{l}}_0$ was dealt with in the previous papers[12, 13], where the dipolar quantization at the SSD point ($\theta \rightarrow \infty$) was directly discussed.

According to Refs.[12, 13], we consider the eigenfunction of \mathfrak{l}_0 ,

$$\mathfrak{l}_0 f_\kappa(z) = -\kappa N_\theta f_\kappa(z), \quad (9)$$

where we have factorized out the normalization factor N_θ for later convenience. Assuming that this eigenfunction is single valued on the complex plane, we have a discrete eigenvalue

$$\kappa N_\theta = n \quad (10)$$

with an integer index n , and the corresponding eigenfunction

$$f_n(z) = A_n \left(\frac{1 - \cosh(2\theta) + \sinh(2\theta)z}{1 + \cosh(2\theta) - \sinh(2\theta)z} \right)^n = A_n (-\tanh(\theta))^n \left(\frac{\sinh(\theta) - \cosh(\theta)z}{\cosh(\theta) - \sinh(\theta)z} \right)^n \quad (11)$$

where A_n is a normalization constant.

On the basis of $f_n(z)$, we next define the differential operator with an index n ,

$$\mathfrak{l}_n = -g(z)f_n(z)\frac{\partial}{\partial z}. \quad (12)$$

Then, a straightforward calculation yields

$$[\mathfrak{l}_n, \mathfrak{l}_{n'}] = (n' - n)g(z)f_n(z)f_{n'}(z)\frac{\partial}{\partial z} \quad (13)$$

If $A_n = (\tanh(\theta))^{-n}$ is adopted, $A_n A_{n'} = A_{n+n'}$ is satisfied. We can then verify $f_n(z)f_{n'}(z) = f_{n+n'}(z)$ with

$$f_n(z) = (-)^n \left(\frac{\sinh(\theta) - \cosh(\theta)z}{\cosh(\theta) - \sinh(\theta)z} \right)^n, \quad (14)$$

and Eq. (13) becomes

$$[\mathfrak{l}_n, \mathfrak{l}_{n'}] = (n - n')\mathfrak{l}_{n+n'}, \quad (15)$$

which is nothing but the classical Virasoro algebra as well.

An interesting point on the eigenfunction $f_n(z)$ is that it contains the identical form to the Möbius transformation of $\text{SL}(2, \mathbb{R})$ for the complex variable z . If $\theta = 0$, $f_n(z) = -z^n$, which reproduces the conventional classical Virasoro algebra of the radial quantization approach. In the SSD limit ($\theta \rightarrow \infty$), on the other hand, $\kappa = n/N_\theta$ becomes continuous, and

$$\begin{aligned} f_\kappa(z) &= \left(\frac{1 - \cosh(2\theta) + \sinh(2\theta)z}{1 + \cosh(2\theta) - \sinh(2\theta)z} \right)^{\kappa N_\theta} \\ &\simeq \left(\frac{1 - \frac{1}{1-z} \frac{1}{N_\theta}}{1 + \frac{1}{1-z} \frac{1}{N_\theta}} \right)^{\kappa N_\theta} \xrightarrow{N_\theta \rightarrow \infty} \exp \left(\frac{2\kappa}{z-1} \right), \end{aligned} \quad (16)$$

which is the same as the kernel function obtained by Ishibashi and Tada[12, 13]. Then, a scaled differential operator $\tilde{\mathfrak{l}}_\kappa \equiv \mathfrak{l}_n/N_\theta$ becomes a generator of the classical Virasoro algebra with the continuous index κ .

A. Möbius coordinate

We briefly analyze the relation of \mathfrak{l}_0 with the Möbius coordinate, which plays an essential role in quantizing \mathfrak{l}_n . As in the usual CFT, we may respectively regard a time development operator and a spatial translation operator as $\mathfrak{l}_0 + \bar{\mathfrak{l}}_0$ and $\mathfrak{l}_0 - \bar{\mathfrak{l}}_0$. On this basis, we can specify

the complex coordinate $\zeta = \tau + is$ with

$$-\frac{\partial}{\partial \tau} = \mathfrak{l}_0 + \bar{\mathfrak{l}}_0, \quad (17)$$

$$-\frac{\partial}{\partial s} = i(\mathfrak{l}_0 - \bar{\mathfrak{l}}_0), \quad (18)$$

which leads us to

$$\zeta = \tau + is = \ln \left(-\tanh(\theta) \frac{\sinh(\theta) - \cosh(\theta)z}{\cosh(\theta) - \sinh(\theta)z} \right) \quad (19)$$

Then, the real part of this equation is reduced to

$$\coth(\theta)e^\tau = \left| \frac{\sinh(\theta) - \cosh(\theta)z}{\cosh(\theta) - \sinh(\theta)z} \right|, \quad (20)$$

where we have assumed $\theta \geq 0$ without loss of generality. Writing $z = x + iy$, we can draw constant- τ contours in the z plane. For $\tau \neq 0$, we have a circle of the radius R and the center $(X, 0)$:

$$(x - X)^2 + y^2 = R^2, \quad (21)$$

where

$$X = \frac{1}{e^{2\tau} - 1} \frac{1}{\sinh(\theta) \cosh(\theta)} + \frac{1}{\tanh(\theta)}, \quad (22)$$

and

$$R = \frac{e^\tau}{|e^{2\tau} - 1|} \frac{1}{\sinh(\theta) \cosh(\theta)}. \quad (23)$$

In Fig. 1, we show a typical time-flow diagram with constant- τ contours for $\theta = 1$. At $\tau = -\infty$, the infinitely small circle is located at $X = \tanh(\theta)$, indicating the source of the time flow. As τ increases to 0, the center position moves toward $X = -\infty$, and, at the same time, the radius R also becomes larger. At $\tau = 0$, the contour (20) reduces to the vertical line of

$$x = 1/\tanh(2\theta), \quad (24)$$

with

$$s = 2 \arctan(\sinh(2\theta)y), \quad (25)$$

which runs over $[-\pi, \pi]$. As τ develops beyond $\tau = 0$, X moves from $X = \infty$ to $1/\tanh(\theta)$, with the radius R shrinking. Thus, the time flow is described by the gradient flow against the constant τ circles, departing from the source of $(\tanh(\theta), 0)$ at $\tau = -\infty$ and arriving at the sink of $(\tanh(\theta), 0)$ at $\tau = \infty$.

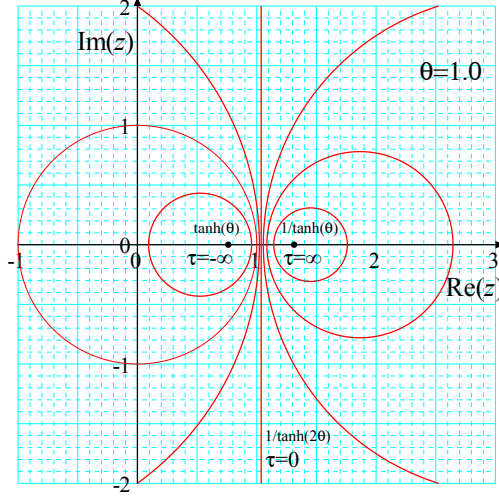


FIG. 1. The time flow in the Möbius coordinate and constant- τ contours for $\theta = 1$.

We next consider the spatial range of s for a fixed τ . Taking the imaginary part of Eq. (19), we obtain

$$s = \arctan \left(\frac{(e^{2\tau} - 1) \sin \phi}{(e^{2\tau} + 1) \cos \phi + 2e^\tau} \right), \quad (26)$$

where we have introduced the parameterization,

$$x = X + R \cos \phi, \quad y = R \sin \phi \quad (27)$$

for a constant- τ circle. An important point is that, if ϕ runs over $[-\pi, \pi]$, s also runs over $[-\pi, \pi]$, independent of τ and θ [17]. Together with (25) for the $\tau = 0$ case, we conclude that the spatial coordinate s always moves in

$$-\pi \leq s \leq \pi, \quad (28)$$

which is a finite strip of the width 2π independent of τ and θ .

III. MÖBIUS QUANTIZATION

In the usual CFT, the stress tensor $T(z)$ (and $\bar{T}(\bar{z})$) is formally expanded by the Laurent series around $z = 0$ as

$$T(z) = \sum_n z^{-n-2} L_n \quad (29)$$

which defines the conventional generator of the Virasoro algebra. In this section, we directly discuss the central extension of the Virasoro algebra on the Möbius coordinate and its relation to L_n .

A. conserved charge and generator

On the basis of the space-time structure of ζ in the previous section, we consider a conserved charge of the stress tensor for the constant- τ contour

$$\begin{aligned}\mathcal{L}_n &= \oint_{\tau} \frac{dz}{2\pi i} g(z) f_n(z) T(z) \\ &= -(-)^n \frac{\sinh(2\theta)}{2 \tanh^n(\theta)} \oint_{\tau} \frac{dz}{2\pi i} \frac{(z - \tanh(\theta))^{n+1}}{(z - 1/\tanh(\theta))^{n-1}} T(z),\end{aligned}\quad (30)$$

where the suffix τ of the integral indicates that the contour integral is performed along the circle of $\tau = \text{const}$ surrounding the time source of $z = 1/\tanh(\theta)$.

The integrand of Eq. (30) has no singularity for $n = 0, \pm 1$, reflecting the global conformal symmetry of $\text{SL}(2, \mathbb{R})$. Then, setting up the contour surrounding $z = 0$ and $1/\tanh(\theta)$, we have

$$\mathcal{L}_0 = \cosh(2\theta) L_0 - \sinh(2\theta) \frac{L_1 + L_{-1}}{2}, \quad (31)$$

which reproduces the Lorentz transformation of Eq. (2). We can also obtain the closed form of \mathcal{L} for $n = \pm 1$,

$$\mathcal{L}_1 = \frac{\sinh(2\theta)}{2} \left(\frac{1}{\tanh(\theta)} L_1 - 2L_0 + \tanh(\theta) L_{-1} \right), \quad (32)$$

$$\mathcal{L}_{-1} = \frac{\sinh(2\theta)}{2} \left(\tanh(\theta) L_1 - 2L_0 + \frac{1}{\tanh(\theta)} L_{-1} \right). \quad (33)$$

Moreover, these operators satisfy the $\text{sl}(2)$ subalgebra

$$[\mathcal{L}_1, \mathcal{L}_{-1}] = 2\mathcal{L}_0, \quad [\mathcal{L}_0, \mathcal{L}_{\pm 1}] = \mp \mathcal{L}_{\pm 1}. \quad (34)$$

For calculation of $n > 1$, the algebraic series expansion with respect to $z \tanh(\theta)$ gives

$$\mathcal{L}_n = -(-)^n \frac{\sinh(2\theta)}{2(\tanh(\theta))^{-n}} \oint \frac{dz}{2\pi i} (1 - z/\tanh(\theta))^{n+1} \left(\sum_{m=0}^{\infty} (z \tanh(\theta))^m \right)^{n-1} \sum_l z^{-l-2} L_l \quad (35)$$

For $n = 2$, for instance, we explicitly have

$$\begin{aligned}\mathcal{L}_2 &= -\frac{\sinh(2\theta)}{2} \left[t^2 L_{-1} + (t^3 - 3t) L_0 + (t^4 - 3t^2 + 3) L_1 + \left(t^5 - 3t^3 + 3t - \frac{1}{t} \right) L_2 \right. \\ &\quad \left. + (t^6 - 3t^4 + 3t^2 - 1) L_3 \cdots \right]\end{aligned}\quad (36)$$

where $t \equiv \tanh(\theta)$. In general, expansion coefficients in front of L_l can be described by a polynomial with respect to $\tanh(\theta)$, the detail of which is presented in Appendix A.

For the analytic continuation of $n \leq -2$, we expand the integrand of (30) around the $z = \infty$, because $z/\tanh(\theta)$ is beyond the convergence radius of the algebraic series. Using $w = 1/z$, we have

$$\mathcal{L}_n = -(-)^n \frac{\sinh(2\theta)}{2 \tanh^n(\theta)} \oint \frac{dw}{2\pi i} (1 - w/\tanh(\theta))^{-n+1} \left(\sum_{m=0}^{\infty} (w \tanh(\theta))^m \right)^{-n-1} \sum_l w^{l-2} L_l \quad (37)$$

For $n = 2$, we have

$$\begin{aligned} \mathcal{L}_{-2} = & -\frac{\sinh(2\theta)}{2} \left[t^2 L_1 + (t^3 - 3t) L_0 + (t^4 - 3t^2 + 3) L_{-1} + \left(t^5 - 3t^3 + 3t - \frac{1}{t} \right) L_{-2} \right. \\ & \left. + (t^6 - 3t^4 + 3t^2 - 1) L_{-3} + \dots \right]. \end{aligned} \quad (38)$$

Here, we make some comments on the above results. First, we can easily see $\mathcal{L}_{\pm 2} \rightarrow L_{\pm 2}$ in the $\theta \rightarrow 0$ limit, implying that \mathcal{L}_n recovers the conventional Virasoro generator of the radial quantization. In the SSD limit ($\theta \rightarrow \infty$), on the other hand, we can see that all of $\mathcal{L}_{\pm 1}$ and $\mathcal{L}_{\pm 2}$ reduce to $\text{const} \times \mathcal{L}_0$. This suggests that \mathcal{L}_n of a finite n collapses to \mathcal{L}_0 in the $\theta \rightarrow \infty$ limit, reflecting $\kappa = n/N_\theta \rightarrow 0$. In other words, a continuous spectrum may be well defined for $n, N_\theta \rightarrow \infty$ with fixing $n/N_\theta = \kappa$, as will be seen in the next subsection.

Next, we discuss the Hermitian conjugate of the generator \mathcal{L}_n . We can see that Eqs. (35) and (37) have the symmetric form with respect to exchanging $n \leftrightarrow -n$. If L_n is equipped with the standard Hermitian conjugate of $L_{-n} = L_n^\dagger$, this symmetric relation of (35) and (37) ensures the Hermiticity of the generator,

$$\mathcal{L}_{-n} = \mathcal{L}_n^\dagger, \quad (39)$$

as long as the series is convergent. Comparing (36) and (38), indeed, we can easily verify $\mathcal{L}_{-2} = \mathcal{L}_2^\dagger$ for $n = 2$. However, it should be remarked that the convergence radii of the geometric series used in Eqs. (35) and (37) require $|\tanh(\theta)| < 1$. Thus, the conventional Hermitian conjugate of Eq. (39) may not be the case just at the SSD point where $\tanh(\theta) = 1$. As was pointed out in Ref. [13], another definition of the Hermitian conjugate may be needed at the SSD point.

B. Virasoro algebra

We next construct the commutator for the generator \mathcal{L}_n . The operator product expansion (OPE) of the stress tensor is given by

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} \dots \quad (40)$$

where c is the central charge. Then, we can directly calculate the commutator with the contour integrals following the time ordering in the Möbius quantization:

$$\begin{aligned}
& [\mathcal{L}_m, \mathcal{L}_n] \\
&= \oint_{C_>} \frac{dz}{2\pi i} g(z) f_m(z) \oint_C \frac{dw}{2\pi i} g(w) f_n(w) T(z) T(w) - \oint_C \frac{dw}{2\pi i} g(w) f_m(w) \oint_{C_<} \frac{dz}{2\pi i} g(z) f_n(z) T(w) T(z) \\
&= \oint_C \frac{dw}{2\pi i} g(w) f_m(w) \oint_{C_w} \frac{dz}{2\pi i} g(z) f_n(z) \left[\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} \dots \right],
\end{aligned}$$

where the integration contours are depicted in Fig. 2. Substituting $g(z)$ and $f_n(z)$ in the above expression and performing the contour integral of z , we obtain

$$\begin{aligned}
& [\mathcal{L}_m, \mathcal{L}_n] \\
&= (-)^{n+m} \frac{\sinh^2(2\theta)}{4 \tanh^{m+n}(\theta)} \int_C \frac{dw}{2\pi i} \left\{ \frac{cn(n-1)(n+1)}{12(\sinh(\theta) \cosh(\theta))^3} \frac{(w - \tanh(\theta))^{n+m-1}}{(w - \frac{1}{\tanh(\theta)})^{n+m+1}} \right. \\
&\quad \left. + \left[2 \left(\frac{(w - \tanh(\theta))^{m+1}}{(w - \frac{1}{\tanh(\theta)})^{m-1}} \right) \left(\frac{(w - \tanh(\theta))^{n+1}}{(w - \frac{1}{\tanh(\theta)})^{n-1}} \right)' - \left(\frac{(w - \tanh(\theta))^{m+1}}{(w - \frac{1}{\tanh(\theta)})^{m-1}} \frac{(w - \tanh(\theta))^{n+1}}{(w - \frac{1}{\tanh(\theta)})^{n-1}} \right)' \right] T(w) \right\},
\end{aligned}$$

where we have used the integration by part for the last term in the square bracket. Then, a straightforward calculation of the w integral yields the Virasoro algebra of the generator \mathcal{L} of the Möbius quantization:

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} + \frac{c}{12}m(m-1)(m+1)\delta_{n+m,0}. \quad (41)$$

Thus, the Lorentz transformation (2) basically provides the same physics as CFT based on the radial quantization. Moreover, this fact indicates that the Möbius quantization can be obtained through the conformal map of the Möbius transformation. We will comment this point in the next section. Nevertheless, we would like to emphasize that the Möbius quantization clarifies an origin of the continuous Virasoro algebra obtained by the dipolar quantization, as follows.

An important viewpoint of Eq. (41) is that the SSD limit, i.e. $N_\theta (= \cosh(2\theta)) \rightarrow \infty$ should correspond to the dipolar quantization by Ishibashi and Tada. Recalling Eqs. (7) and (10), we define the scaled spectrum and generator as

$$n/N_\theta \rightarrow \kappa, \quad \mathcal{L}_n/N_\theta \rightarrow \tilde{\mathcal{L}}_\kappa, \quad N_\theta \delta_{n,0} \rightarrow \delta(\kappa). \quad (42)$$

In the $N_\theta \rightarrow \infty$ limit, then, κ becomes the continuous index and Eq. (41) is reduced to

$$[\tilde{\mathcal{L}}_\kappa, \tilde{\mathcal{L}}_{\kappa'}] = (\kappa - \kappa')\tilde{\mathcal{L}}_{\kappa+\kappa'} + \frac{c}{12}\kappa^3\delta(\kappa + \kappa'), \quad (43)$$

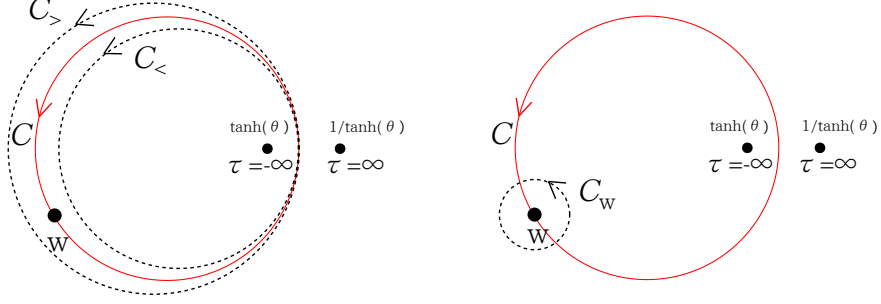


FIG. 2. The contours in the left panel can be deformed to those in the right panel.

which is so called continuous Virasoro algebra. In this sense, the continuous Virasoro algebra of the dipolar quantization can be interpreted as a continuum limit of the Virasoro algebra for the scaled generator.

C. Vacuum

In the usual CFT based on the radial quantization, the vacuum is defined as

$$|0\rangle = I(z=0)|0\rangle \quad \text{and} \quad L_n|0\rangle = 0, \quad \text{for } n \geq -1 \quad (44)$$

where $I(z)$ is the identity operator. Then, the Hermiticity $L_n^\dagger = L_{-n}$ ensures

$$\langle 0|L_n = 0, \quad \text{for } n \leq 1 \quad (45)$$

For the case of Möbius quantization, we may define the vacuum as

$$|0\rangle_\theta = I(z = \tanh(\theta))|0\rangle_\theta \quad \text{and} \quad \mathcal{L}_n|0\rangle_\theta = 0, \quad \text{for } n \geq -1, \quad (46)$$

where $|0\rangle_\theta$ denotes the vacuum for the Möbius quantization. The Hermiticity of Eq. (39) gives

$${}_\theta\langle 0|\mathcal{L}_n = 0, \quad \text{for } n \leq 1 \quad (47)$$

Here, we should recall that Eqs. (31), (32), (33) and (35) imply that \mathcal{L}_n for $n \geq -1$ can be expanded by only L_n with $n \geq -1$. Thus, we trivially have

$$|0\rangle_\theta = |0\rangle, \quad (48)$$

which is also a consequence of the $SL(2, \mathbb{R})$ invariance of the SSD Hamiltonian.

D. Primary field

We next discuss the primary field. In general, for a conformal transformation $z \rightarrow w(z)$, a primary field of a scaling dimension h is defined by

$$\phi'(w) = \left(\frac{dz}{dw} \right)^h \phi(z), \quad (49)$$

which leads us to the commutator

$$[L_n, \phi_h(z)] = z^{n+1} \partial_z \phi_h(z) + (n+1) h z^n \phi_h(z), \quad (50)$$

or, equivalently, the OPE

$$T(z) \phi_h(w) = \frac{h \phi_h(w)}{(z-w)^2} - \frac{\partial_w \phi_h(w)}{z-w} \dots \quad (51)$$

In particular, the primary field specifies the highest weight state as

$$L_0 \phi_h(x) = h \phi_h(x) \quad \text{or} \quad L_0 |h\rangle = h |h\rangle, \quad (52)$$

where $|h\rangle \equiv \phi_h(0)|0\rangle$.

For the Möbius quantization, we have the commutator

$$\begin{aligned} [\mathcal{L}_n, \phi_h(z)] &= \oint_z \frac{dw}{2\pi i} g(w) f_n(w) T(w) \phi_h(z) \\ &= -(-)^n \frac{\sinh(2\theta)}{2(\tanh(\theta))^n} \left[h \left(\frac{(z - \tanh(\theta))^{n+1}}{(z - \frac{1}{\tanh(\theta)})^{n-1}} \right)' \phi_h(z) - \left(\frac{(z - \tanh(\theta))^{n+1}}{(z - \frac{1}{\tanh(\theta)})^{n-1}} \right) \partial_z \phi_h(z) \right] \end{aligned} \quad (53)$$

where we have used the OPE (51). Particularly for $n = 0$, we have explicitly

$$\begin{aligned} [\mathcal{L}_0, \phi_h(z)] &= -\frac{\sinh(2\theta)}{2} \left[h \left((z - \tanh(\theta)) + (z - \frac{1}{\tanh(\theta)}) \right) \phi_h(z) \right. \\ &\quad \left. - (z - \tanh(\theta)) (z - \frac{1}{\tanh(\theta)}) \partial_z \phi_h(z) \right] \end{aligned} \quad (54)$$

If we define the primary state for the Möbius quantization as

$$|h\rangle_\theta = \phi_h(z = \tanh(\theta)) |0\rangle, \quad (55)$$

we can easily see

$$\mathcal{L}_0 |h\rangle_\theta = h |h\rangle_\theta. \quad (56)$$

From Eq. (53), moreover, we straightforwardly read,

$$\mathcal{L}_n |h\rangle_\theta = 0, \quad \text{for } n \geq 1. \quad (57)$$

For $n \leq -1$, on the other hand, \mathcal{L}_n generates a new state $\mathcal{L}_n|h\rangle_\theta$, as in the usual CFT. Using the commutator (41) for \mathcal{L}_n , we can thus construct the representation of the Virasoro algebra for the primary state (55).

The relation between $|h\rangle_\theta$ and $|h\rangle$ can be directly seen through the conventional Virasoro generator by the radial quantization; Since L_{-1} is the translation operator in the z plane, we have

$$\phi_h(\alpha) = e^{\alpha L_{-1}} \phi_h(0) e^{-\alpha L_{-1}}. \quad (58)$$

Then, Eq. (55) is rewritten as

$$|h\rangle_\theta = e^{\alpha L_{-1}} \phi_h(0) e^{-\alpha L_{-1}} |0\rangle = e^{\alpha L_{-1}} \phi_h(0) |0\rangle = e^{\alpha L_{-1}} |h\rangle \quad (59)$$

with $\alpha = \tanh(\theta)$. Using the commutators in Appendix B, we can calculate

$$\begin{aligned} & \left(\cosh(2\theta) L_0 - \frac{\sinh(2\theta)}{2} (L_{-1} + L_1) \right) e^{\alpha L_{-1}} |h\rangle \\ &= \left[h (\cosh(2\theta) - \alpha \sinh(2\theta)) + \left(\alpha \cosh(2\theta) - \frac{\sinh(2\theta)}{2} (\alpha^2 + 1) \right) L_{-1} \right] e^{\alpha L_{-1}} |h\rangle, \end{aligned} \quad (60)$$

which enables us to directly verify

$$\mathcal{L}_0 e^{\tanh(\theta) L_{-1}} |h\rangle = h e^{\tanh(\theta) L_{-1}} |h\rangle. \quad (61)$$

This result is of course consistent with Eq. (56). Moreover, the norm of $|h\rangle_\theta$ can be evaluated as

$${}_\theta \langle h|h \rangle_\theta = \langle h| e^{\alpha L_1} e^{\alpha L_{-1}} |h\rangle = 1 + \sum_{n=1}^{\infty} B_n, \quad (62)$$

with

$$B_n = \frac{\alpha^{2n}}{n!} (n-1+2h)(n-2+2h) \cdots (1+2h)(2h). \quad (63)$$

If $\alpha = \tanh(\theta) < 1$,

$$\left| \frac{B_{n+1}}{B_n} \right| = \alpha^2 \left(1 + \frac{2h-1}{n+1} \right) < 1 \quad \text{for } n > \exists n_c. \quad (64)$$

Thus, the series of Eq. (62) is convergent for $\tanh(\theta) < 1$, implying that $|h\rangle_\theta$ is normalizable. In the SSD limit, however, the series is divergent[11]. This suggests that it may be difficult to construct the primary field just at the SSD point within the present approach where the series of (35) and (37) are also divergent. In this sense, the primary field for the dipolar quantization is a nontrivial problem.

IV. THE CONFORMAL MAPPING APPROACH

In the previous section, we have quantized CFT on the Möbius coordinate, which yields the Virasoro algebra of the same form as that of the radial quantization. Of course, this is a natural consequence of the $SL(2, \mathbb{R})$ symmetry in CFT, which suggests a more direct approach to obtain the Virasoro algebra of \mathcal{L} . Let us consider the Möbius-type conformal map of $SL(2, \mathbb{R})$.

$$w = -\frac{\sinh(\theta) - \cosh(\theta)z}{\cosh(\theta) - \sinh(\theta)z} \quad (65)$$

We then define the Virasoro generator in the mapped coordinate(w -plane) as

$$\mathcal{L}_n \equiv \int \frac{dw}{2\pi i} w^{n+1} T(w). \quad (66)$$

Now, we rewrite \mathcal{L}_n in terms of the z -plane. Recall that, for the conformal map (65), the stress tensor transforms as

$$T(w) = \left(\frac{dz}{dw}\right)^2 T(z) = (\cosh(\theta) - \sinh(\theta)z)^4 T(z). \quad (67)$$

Thus, we have

$$\mathcal{L}_n = \int \frac{dz}{2\pi i} \frac{dw}{dz} w^{n+1} \left(\frac{dz}{dw}\right)^2 T(z) = -(-)^n \frac{\sinh(2\theta)}{2(\tanh(\theta))^n} \int \frac{dz}{2\pi i} \frac{(z - \tanh(\theta))^{n+1}}{(z - 1/\tanh(\theta))^{n-1}} T(z), \quad (68)$$

which is consistent with Eq. (30). We can therefore obtain the results of the Möbius quantization, without passing through complicated analysis of the Möbius coordinate. However, the Möbius quantization approach interpolating between the radial and dipolar quantizations was essential to reveal the continuum limit of the Virasoro algebra.

V. DISCUSSION

We have discussed the two-dimensional CFT in terms of the Möbius quantization, which bridges the radial quantization and the dipolar quantization. An essential feature of the Möbius coordinate is that the source and sink of the time flow are respectively located at $z = \tanh(\theta)$ and $1/\tanh(\theta)$, and they move toward $z = 1$, as the deformation parameter θ increases. As far as θ is finite, then, we have constructed the Virasoro algebra for the generator \mathcal{L}_n defined by Eq. (30), consistent with the result of the conformal mapping of $SL(2, \mathbb{R})$. In the SSD ($\theta \rightarrow \infty$) limit, the locations of the source and sink finally fuse with

each other at $z = 1$, which corresponds to the dipolar quantization, where the continuous Virasoro algebra emerges for the scaled generator $\tilde{\mathcal{L}}_n$ defined by Eq. (42).

An important implication of the Möbius quantization approach is that the continuous Virasoro algebra for $\theta \rightarrow \infty$ can be viewed as a continuum limit of the conventional Virasoro algebra. As in the previous works, it is natural to deal with the scaled Hamiltonian $\tilde{H} = \tilde{\mathcal{L}}_0 + \bar{\tilde{\mathcal{L}}}_0$ at the SSD point, so that \mathfrak{l}_0 and $\bar{\mathfrak{l}}_0$ in Eqs. (17) and (18) should be replaced by $\tilde{\mathfrak{l}}_0$ and $\bar{\tilde{\mathfrak{l}}}_0$. Then, the $\tilde{\zeta}$ coordinate involves the scale factor N_θ :

$$\tilde{\zeta} = \tilde{\tau} + i\tilde{s} = N_\theta \ln \left(-\tanh(\theta) \frac{\sinh(\theta) - \cosh(\theta)z}{\cosh(\theta) - \sinh(\theta)z} \right). \quad (69)$$

which implies that the scale of the spatial coordinate is also modified by N_θ , and the range of \tilde{s} becomes

$$-\pi N_\theta \leq \tilde{s} \leq \pi N_\theta, \quad (70)$$

in contrast to (28). Thus, the scaled Virasoro generator $\tilde{\mathcal{L}}_n$ has the θ -dependent circumference, $2\pi N_\theta$, which diverges as $\theta \rightarrow \infty$. This is the infinite circumference limit of the dipolar quantization in Refs. [12, 13].

We can also discuss the role of the scale factor N_θ in analogy with the finite-size scaling of CFT. For the radial quantization, the finite-size-scaling analysis for a cylinder is obtained through the conformal mapping $z' = \frac{\ell}{2\pi} \ln z$, where ℓ is a free parameter representing the strip width of the cylinder[18]. For the scaled Möbius coordinate (69), we can rewrite $\tilde{\zeta}$ as

$$\tilde{\zeta} = \frac{\ell_\theta}{2\pi} \ln w + C, \quad (71)$$

with w of Eq. (65) and $\ell_\theta \equiv 2\pi N_\theta$. Since $C \simeq -1$ for $\theta \gg 1$, this relation can be viewed as a finite-size scaling for the w coordinate. This suggests that, as ℓ_θ increases, the region of the energy scale exhibiting the conformal-tower spectrum for the scaled Hamiltonian $\tilde{H} = \tilde{\mathcal{L}}_0 + \bar{\tilde{\mathcal{L}}}_0$ is expected to collapse into the infinitely small energy scale above the ground state. The numerical finite-size-scaling analysis of the free fermion system at the SSD point actually observed the $1/L^2$ size dependence of the excitation spectrum[9], instead of the universal $1/L$ -dependent spectrum of CFT. Moreover, the analytic calculation with help of SUSY quantum mechanics for the non-relativistic free fermion provides the exact $1/L^2$ dependence of the spectrum.[6] By contrast, the low-energy spectrum of the unscaled Hamiltonian $\mathcal{H} = \mathcal{L}_0 + \bar{\mathcal{L}}_0$ may maintain the usual conformal spectrum, where the scale of \mathcal{H} is magnified by the huge scale factor N_θ .

As was seen above, some properties of the spectrum of the scaled generator $\tilde{\mathcal{L}}$ can be understood in the analogy with the finite-size scaling. Also, the continuous Virasoro algebra (43) for the scaled generator $\tilde{\mathcal{L}}$ is basically consistent with the dipolar quantization. However, we should mention that there remain some difficulties in the SSD limit. ℓ_θ in Eq. (71) is not a free parameter but the scale factor embedded in the Lorentz transformation, and thus $\tilde{\zeta}$ has the limit, $\tilde{\zeta} = \frac{2}{z-1}$ for $\theta \rightarrow \infty$. This function corresponds to the mapping of $u = e^{\tilde{\zeta}} = \exp(\frac{2}{z-1})$ containing the essential singularity at $z = 1$, which causes peculiar behaviors in the dipolar quantization.[11–13]. Accordingly, the primary field (55) is unnormalizable at the SSD point, and the Hermitian conjugate (39) is inconsistent with that defined in Ref. [13]. A significant point of the SSD limit is that the source and sink of the time flow fuse at $z = 1$, which never occurs for the case of the usual CFT where $\tau = -\infty$ is assumed to be isolated in the complex plane. Further investigations are clearly required for understanding the SSD and the continuous Virasoro algebra. We then believe that the relation (42) provides a solid footing to address the SSD limit. In addition, it should be noted that a connection to similar quantization approaches based on the global conformal symmetry[19], such as NS quantization[20], is also interesting problem.

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- [1] A. Gendiar, R. Krcmar, and T. Nishino, “Spherical Deformation for One-dimensional Quantum Systems” *Prog. Theor. Phys.* **122**, 953 (2009); **123**, 393 (2010).
 - [2] T. Hikihara and T. Nishino, “Connecting distant ends of one-dimensional critical systems by a sine-square deformation”, *Phys. Rev. B* **83**, 060414 (2011).
 - [3] A. Gendiar, M. Daniška, Y. Lee, and T. Nishino, “Suppression of finite-size effects in one-dimensional correlated systems” *Phys. Rev. A* **83**, 052118 (2011).

- [4] H. Katsura, “Exact ground state of the sine-square deformed XY spin chain”, J. Phys. A: Math. Theor. **44**, 252001 (2011).
- [5] I. Maruyama, H. Katsura, and T. Hikihara, “Sine-square deformation of free fermion systems in one and higher dimensions”, Phys. Rev. B **84**, 165132 (2011).
- [6] K. Okunishi and H. Katsura, “Sine-square deformation and supersymmetric quantum mechanics” J. Phys. A: Math. Theor. **48**, 445208 (2015)
- [7] N. Shibata and C. Hotta, “Boundary effects in the density-matrix renormalization group calculation” Phys. Rev. B **84**, 115116 (2011).
- [8] C. Hotta and S. Nishimoto, “Grand Canonical Finite size numerical approaches : a route to measuring bulk properties under applied field”, Phys. Rev. B **86**, 041108 (2012).
- [9] C. Hotta, S. Nishimoto, and N. Shibata, “Grand canonical finite size numerical approaches in one and two dimensions: Real space energy renormalization and edge state generation”, Phys. Rev. B **87**, 115128 (2013).
- [10] H. Katsura, “Sine-square deformation of solvable spin chains and conformal field theories” J. Phys. A: Math. Theor. **45**, 115003 (2012).
- [11] T. Tada, “Sine-Square Deformation and its Relevance to String Theory”, Mod.Phys.Lett. **A30**, 1550092 (2015), arXiv:1404.6343.
- [12] N. Ishibashi and T. Tada, “Infinite circumference limit of conformal field theory”, J. Phys. A: Math. Theor. **48** (2015) 315402.
- [13] N. Ishibashi and T. Tada, “Dipolar quantization and the infinite circumference limit of two-dimensional conformal field theories”, arXiv:1602.01190 .
- [14] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, “Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory,” Nucl. Phys. B **241**, 333 (1984).
- [15] A parameterized Hamiltonian for the SSD was proposed for the free fermion chain in Ref. [5].
- [16] Hereafter, the antiholomorphic part of generators will not be shown explicitly, except for some special cases. However we should always suppose the antiholomorphic part parallel to the holomorphic part.
- [17] Thus, s is not the principal value of Arctan .
- [18] J.L. Cardy, “Conformal invariance and universality in finite-size scaling”, J.Phys. **A17**, L385 (1984); “Operator content of two-dimensional conformally invariant theories” Nucl.Phys. **B270**, 186 (1986).

- [19] M. Lüscher and G. Mack, “Global conformal invariance in quantum field theory”, Comm. Math. Phys. **41**, 203 (1975)
- [20] S. Rychkov, “EPFL Lectures on Conformal Field Theory in $D \geq 3$ Dimensions”, arXiv:1601.05000.

Appendix A: expansion coefficients of \mathcal{L}_n

The Virasoro generator \mathcal{L}_n of the Möbius quantization can be represented by a series of the conventional Virasoro generator L_l of the radial quantization. We assume $n > 1$ for Eq. (30). Using $t = \tanh(\theta) (< 1)$ for simplicity, we have

$$\begin{aligned}\mathcal{L}_n &= -(-)^n \frac{\sinh(2\theta)}{2t^n} \sum_l L_l \oint_{\tau} \frac{dz}{2\pi i} \frac{(z-t)^{n+1}}{(z-1/t)^{n-1}} z^{-l-2} \\ &= -(-)^n \frac{\sinh(2\theta)}{2} \sum_l C_l^n(t) L_l,\end{aligned}\tag{A1}$$

where the expansion coefficient is defined by

$$C_l^n(t) \equiv t^{n-l-1} \oint_{(0,1)} \frac{dy}{2\pi i} y^{-l-2} (1-y)^{n+1} (1-t^2 y)^{-n+1}.\tag{A2}$$

The integration path is the contour surrounding $y = 0$ and 1 in the complex y plane. The pole in the contour is located only at $y = 0$, if $n \geq -1$. Using Goursat’s formula, then, we have

$$C_l^n(t) = t^{n-l-1} \frac{1}{(l+1)!} \frac{d^{l+1}}{dy^{l+1}} (1-y)^{n+1} (1-t^2 y)^{-n+1} \Big|_{y=0},\tag{A3}$$

which is a polynomial of t . For $n > 1$, moreover, we can represent $C_l^n(t)$ with use of Gauss’s hypergeometric function:

$$C_l^n(t) = \begin{cases} (-)^{l+1} \frac{(n+1)!}{(l+1)!(n-l)!} F(n-1, -l-1; n-l+1; t^2) t^{n-l-1} & n \geq l \geq -1 \\ (-)^{n+1} \frac{(l-2)!}{(n-2)!(l-n)!} F(-n-1, l-1; l-n+1; t^2) t^{l-n-1} & n < l \end{cases},\tag{A4}$$

which is equivalent to Eq. (A3). If $n = 2$, the above formula reproduces Eq. (36). For the case of $n < -1$, \mathcal{L}_n can be attributed to Eq. (A2) with $n \rightarrow -n$ after changing the variable $w = 1/z$, as in Eq. (37).

Appendix B: useful commutators

For the Virasoro generators L_0 and $L_{\pm 1}$, we have the following commutators,

$$[L_0, (L_{-1})^n] = n(L_{-1})^n, \quad (\text{B1})$$

$$[L_0, e^{aL_{-1}}] = aL_{-1}e^{aL_{-1}}, \quad (\text{B2})$$

$$[L_1, (L_{-1})^n] = n(n-1)(L_{-1})^{n-1} + 2n(L_{-1})^{n-1}L_0, \quad (\text{B3})$$

$$[L_1, e^{aL_{-1}}] = a^2L_{-1}e^{aL_{-1}} + 2ae^{aL_{-1}}L_0, \quad (\text{B4})$$

which are useful for calculations in Sec. III